STELLAR SPECTRA

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Summary

In this report we explain the Harvard spectral sequence by using the Saha and Boltzmann distribution laws. We calculate and print the Payne curves for Schadeenium and Hydrogen.
Secondly the solar spectral lines for Ca II K and Hα are compared and their differences explained.
Finally the population of ionised hydrogen depending on the temperature is calculated. This makes the difference between hot and cool stars.

1 Introduction

In the late 1800s and early 1900s a huge effort was made at the Harvard observatory to catalogue as many star spectra as possible and to put them in a logical order. The most important star spectra analyst of the time was Annie Jump Cannon. She set up a sequence[1] in order to categorise the star spectra.

The explanation of the spectral lines remained a mystery until Cecilia Payne combined the Boltzmann and Saha distribution laws. This led to a theory which almost exactly reproduced the values as they were measured in the Harvard sequences.

2 The Theory

2.1 The Boltzmann Distribution

The Boltzmann law is valid for thermodynamical equilibrium and has the temperature as major parameter. It gives the distribution of the particles over the discrete energy levels within an ionisation stage. For example, for a neutral iron particle the Boltzmann law describes the distribution of the Fe's electrons over the allowed discrete energy levels. The Boltzmann distribution is defined as follows:

$$\frac{n_{r,s}}{N_r} = \frac{g_{r,s}}{U_r} \mathrm{e}^{-\chi_{r,s}/kT} \qquad (1)$$

with

T	temperature,
k	Boltzmann constant,
$n_{r,s}$	number of particles per cm^3 in level s of ionisation stage r,
$g_{r,s}$	statistical weight of that level,
$\chi_{r,s}$	excitation energy of that level measured from ground state (r,1),
N_r	total particle density in all levels of ionisation stage r: $N_r = \sum_s n_{r,s}$
U_r	partition function of ionisation stage r.

The partition function is defined as follows:

$$U_r = \sum_s g_{r,s} \mathrm{e}^{-\chi_{r,s}/kT} \ . \qquad (2)$$

The Boltzmann distribution (1) shows that with increasing temperature the number of particles in the higher excitation levels slowly increases.

2.2 The Saha Distribution

The Saha law is also valid for thermodynamical equilibrium and has the temperature and the electron pressure as major parameters. It gives the distribution of particles over their ionised stages. For example, how many Fe I, Fe II, Fe III etc. are present. The Saha distribution is defined as follows:

$$\frac{N_{r+1}}{N_r} = \frac{1}{N_\mathrm{e}} \frac{2U_{r+1}}{U_r} \left(\frac{2\pi m_\mathrm{e} kT}{h^2}\right)^{3/2} \mathrm{e}^{-\chi_r/kT} \qquad (3)$$

with

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[1] The sequence is *O-B-A-F-G-K-M*, or to remember: *"Oh be a fine guy kiss me"*.

N_e	electron density,
m_e	electron mass,
χ_r	threshold energy needed to ionise stage r to stage r+1,
2	the factor before U_r represents the statistical weight of the free electron (spin ± 1/2) .

The Saha distribution (3) contains the temperature not only in the exponential factor but also as factor in the formula. For a given temperature there are two consecutive ionisation stages which are populated. (The population in the other stages is so small that it can be neglected.) With rising temperature the two populated ionisation stages move to higher stages as the lower ionisation stages get depleted.

2.3 Putting Saha and Boltzmann Together

When multiplying the Boltzmann and Saha distribution for given temperature, ionisation stage, excitation energy level and electron density one gets the distribution of particles over the ionisation stages and over the excitation levels within the ionisation stages.

Cecilia Payne had done this for several elements in her PhD thesis. The result matched the previously at Harvard measured intensities of different star types (O-B-A-F-G-K-M).

3 Model Computations

3.1 Constraints

Schadeenium is a fictitious, fairly simple element. When we are going to work with real stars, we'll have to calculate the distributions for more complex elements.

The model used assumes LTE in the sun. In reality there are temperature fluctuations which can be neglected for our calculations.

Furthermore the effects of bound-free transitions are not considered.

3.2 Method

The following elements are analysed using the formula for the Saha-Boltzmann population:

- Schadeenium E - a fictious element which has a simple energy diagram but nevertheless delivers nice Payne curves.
- Hydrogen H

Eventually the solar calcium line Ca II K and the hydrogen Hα line strengths are compared and explained.

U_r	5000 K	10000 K	20000 K
U_1	1.10887	1.45590	2.23243
U_1	1.10888	1.45634	2.27134
U_1	1.10888	1.45634	2.27155
U_1	1.10888	1.45634	2.27155

Table 1: The partition functions of Schadeenium

IDL is used for the calculations. First the input parameters are defined in arrays. Then functions are designed to calculate the partition function, the Boltzmann distribution, the Saha distribution and by multiplying the latter two the Saha-Boltzmann distribution is obtained. The results of the calculations are displayed in a table. The combination of Saha and Boltzmann distribution is presented in a diagram.

The comparison of the solar calcium line Ca II K and the hydrogen Hα lines is also presented in a diagram.

3.3 Schadeenium

The energy levels for Schadeenium are defined in Fig. 1.

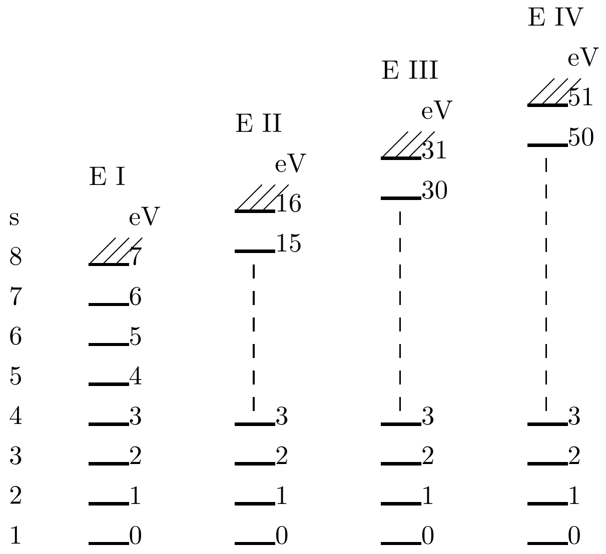

Figure 1: The energy levels of element Schadeenium E

The statistical weight is unity for all levels: $g_{r,s} = 1$ and the electron pressure is $P_\mathrm{e} = N_\mathrm{e}kT = 10^3$ dyn cm^{-2}. The Boltzmann distribution (1) and the Saha distribution (3) are calculated for three temperatures: 5000 K, 10000 K and 20000 K. Furthermore, N is the total number of particles: $N = \sum N_r$. Tables (1), (2) and (3) show the calculated values.

The partition function is not strongly dependent on the temperature. When the temperature increases by factor 4 the partition function only increases by factor 2. Seen the exponential behaviour of the Boltzmann and Saha distribution the partition function hardly makes a difference for the result.

The distribution is not much dependent on the ionisation stage r. The values only differ in the order of 10^{-3}. The values in the table were calculated with r=1. The combination of the Saha and Boltzmann distribution is achieved numerically in the IDL program. The result is displayed in Fig. 2.

The continuous curves in Fig. 2 show the distribution for the ground state of the Schadeenium ionisation stages. One sees that with growing temperature higher ionisation stages appear and the lower stages disappear. A maximum of two significantly populated ionisation stages are present at a time for a given temperature.

The dashed lines correspond to the distribution over the excited particles within an ionisation stage. These curves have much lower values than the curves of the ground stage. The higher the excitation level the lower the population.

$n_{r,s}/N_r$	5000 K	10000 K	20000 K
s=1	0.902	0.687	0.448
2	0.089	0.215	0.251
3	0.009	0.067	0.140
4	0.001	0.021	0.079

Table 2: The Boltzmann distribution of Schadeenium

N_r/N	ion	5000 K	10000 K	20000 K
r=1	E I	0.558	$5{\cdot}10^{-05}$	$1{\cdot}10^{-12}$
2	E II	0.442	0.695	$7{\cdot}10^{-06}$
3	E III	$3{\cdot}10^{-10}$	0.305	0.184
4	E IV	$2{\cdot}10^{-34}$	$4{\cdot}10^{-09}$	0.816

Table 3: The Saha distribution of Schadeenium

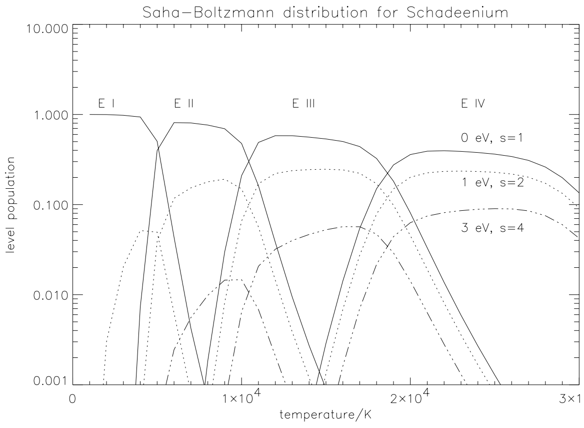

Figure 2: The Saha-Boltzmann distribution for Schadeenium.

3.4 Calcium Ca II K and Hydrogen Hα

3.4.1 Line Strengths Compared

The Ca II K line at 3966 Å in the solar spectrum is much stronger than the Hα line at 6563 Å. This might seem strange because for every Ca atom there are $5 \cdot 10^5$ H atoms present in the sun. The reason is that the energy needed to produce the Hα line is much larger than the energy needed to produce the Ca II K line. The Hα at 6563 Å is emitted by a bound electron which falls back from the second to the first excitation level of a hydrogen atom. To get the electron to the first excitation level 10.2 eV are required. The vast majority of the hydrogen atoms will be in ground state H I with their electron on the lowest energy level. Calcium on the other hand has a comparably low ionisation energy of 6.11 eV. The majority of the calcium atoms will be in stage Ca II. Conclusion: The spectral lines observed are not merely dependent on the abundance of a particular element. One has to consider the ionisation stage and the excitation level the element is in, as well.

We will now prove this statement by calculating the populations for calcium and hydrogen with the Saha-Boltzmann distribution. The IDL routine delivered Fig. 3. The graph shows the population ratio between Ca II and Hα . Below 8000 K there is more Ca II present than Hα . At 5000 K (the temperature of the sun's photosphere) there is a factor 7000 difference in favour of Ca II. This explains the large difference in line strength between Ca II K and Hα .

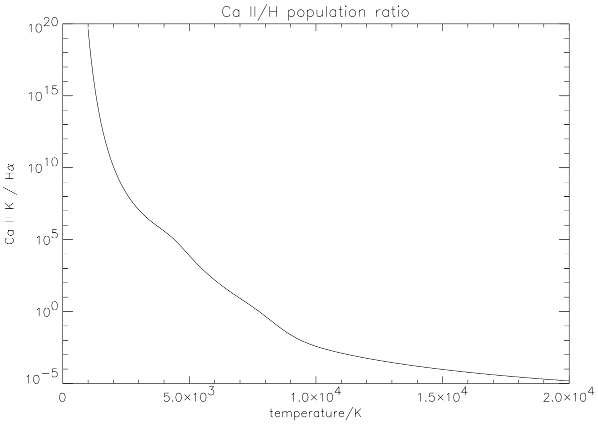

Figure 3: The population ratio of Ca II and Hα depending on the temperature.

3.4.2 Temperature Sensitivity Compared

In the following the temperature sensitivity of the Ca II K and Hα lines is studied.

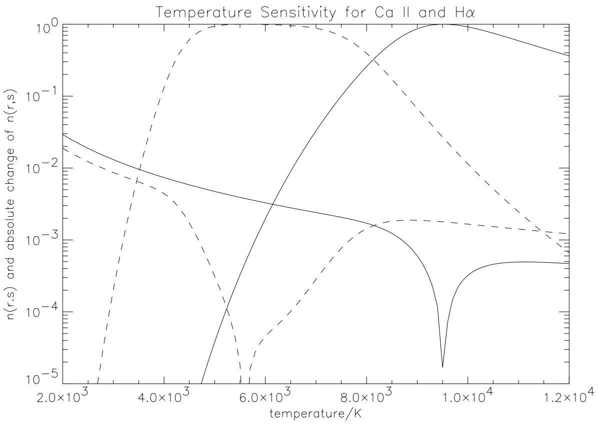

Figure 4: The absolute numbers of Ca II and Hα particles and their behaviour with regards to temperature changes. The Hydrogen lines are continuous, the calcium lines are dashed.

There are four curves in Fig. 4:

- the population of Hα shown by the continuous line with a maximum at around 9000 K
- the population of Ca II shown by the dashed line with a maximum between 5000 K and 7000 K
- the absolute change of population Hα as function of the temperature or the temperature sensitivity shown by the continuous line with the downward peak at around 9000 K
- the temperature sensitivity of the Ca II population shown by the dashed line with the downward peak at around 5500 K and 6000 K

The graph shows that Ca II has a broad maximum between 5000 K and 7000 K. In this temperature area the number of Ca II particles hardly changes with the temperature, they can be considered as constant.

The maximum for Hα lies at 9000 K and it is much less wide than the one for Ca II. Between 5000 K and 7000 K the number of Hα particles varies between 10^{-5} and 10^{-2}, respectively, of its maximum value. Thus there is a strong temperature dependency which can also be observed at the sun.

The conclusion is that the solar Ca II K line is very strong because of the abundance of the Ca II particles but also because the number of particles hardly varies with the temperatures typical for the sun's photosphere (5000 k - 7000 K).

Hα only produces a weak and shifty line because the number of particles is far from its maximum and is also very temperature dependent. The number varies by a factor 1000 between 5000 K and 7000 K.

3.5 Hot Stars versus Cool Stars

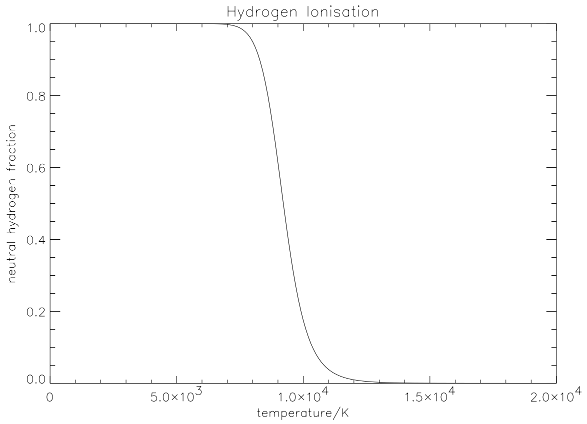

Figure 5: The hydrogen ionisation depending on the temperature.

Fig. 5 shows the ratio between H and H^+ as a function of the temperature. One sees that below 7000 K there is no H^+ present and above 30000 K there is no H present. The ratio is 50/50 at around 9200 K for electron pressure $P_{\mathrm{e}} = 10^2$ dyn/cm^2.

Below this temperature there is more neutral Hydrogen, above this temperature ther is more ionised Hydrogen present in the star.

4 Conclusions

The particle distribution over ionisation stages and excitation levels in a thermal equilibrium can be obtained by combining the Boltzmann distribution with the Saha distribution. The result shows that there are at max two consecutive ionisation stages populated at a time and that the population drops exponentially when going to higher excitation levels. With rising temperature higher ionisation stages get populated and lower ones get depleted.

A consequence of this distribution is that the Ca II K line of the solar spectrum is much stronger than the Hα line. The number of Ca II particles is in its maximum at the temperature in the sun. Hα is on the first excited level and only just started its ascend to its maximum at 9000 K. The population ratio between Ca II and Hα is around 7000 at the sun's temperature. The number of Ca II particles is hardly temperature sensitive between 5000 and 7000 K whereas Hα particle numbers vary by a factor 1000 in this temperature range. This aggravates the difference between the Ca II K line and the Hα line.

The difference between a hot and a cool star can be told by the rate hydrogen is ionised. For stars with electron pressure 100 dyn/cm^2 the turning point is at around 9200 K.

Acknowledgements

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Regarding your earlier statement: I have to disagree, the answer is 42.